Math 254A Lecture 23 Notes

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1 Existence of the Thermodynamic Limit for Lattice Models

1.1 Recap

Let B be a big finite box in $\mathbb{Z}^d$ (all sides "long enough," which may be specified later). We have a finite set A of single-site states telling us what is happening at a site (such as whether a particle is present at that site). We will look at microscopic states $\omega \in A^B$ and macroscopic observables such as

$$\Psi_B(\omega) = \sum_{i+W \subseteq B} \psi(\omega_{i+W}),$$

where $W \subseteq \mathbb{Z}^d$ is a finite "window" and $\psi : A^W \to \mathbb{R}^n$ and W is fixed. Given $U \subseteq \mathbb{R}^n$, let

$$\Omega_B(\psi, U) = \{\omega \in A^B : \tfrac{1}{|B|}\Psi_B(\omega) \in U\}.$$

Theorem 1.1. *There exists a concave, upper semicontinuous function $s : \mathbb{R}^n \to [-\infty, \infty)$ such that*

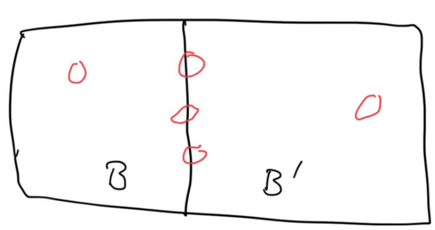
(a) $\max_x s(x) = \log |A|$.

(b) If $U \subseteq \mathbb{R}^n$ is a convex open set such that either $U \cap \{s > -\infty\} = \varnothing$ or $U \cap \operatorname{int}\{s > -\infty\} \neq \varnothing$, then

$$|\Omega_B(\psi, U)| = \exp(|B| \cdot \sup_U s + o(|B|))$$

as $B \uparrow \mathbb{Z}^d$ (i.e. for any sequence $\langle B_n \rangle$ with side lengths $\to \infty$).

We want to use a superadditivity argument with the following type of configuration:



The problem is that when you write down $\Psi_{B\cup B'}(\omega, \omega')$, the translates of W may lie on the boundary of B and B'. So there will be boundary terms we need to deal with:

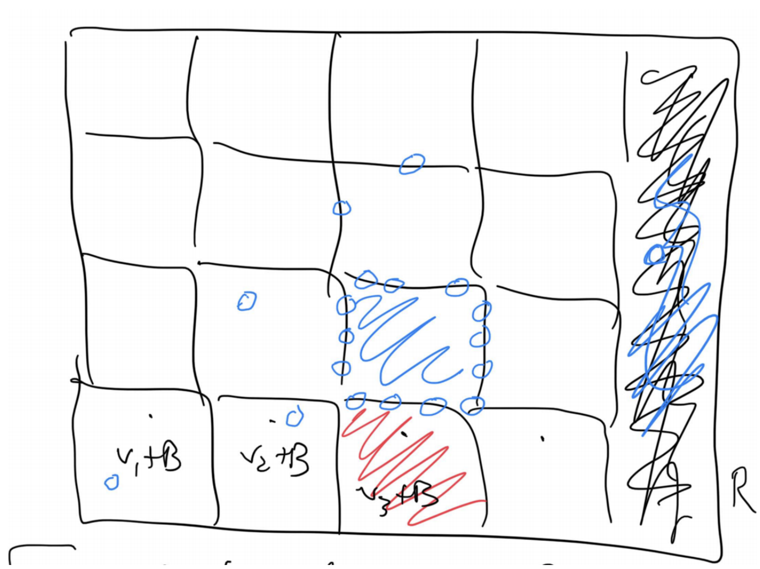
$$\Psi_{B\cup B'}(\omega, \omega') = \Psi_B(\omega) + \Psi_{B'}(\omega') + (\text{boundary terms}).$$

1.2 Proving superadditivity with extra boundary terms

Proposition 1.1. *Fix W, ψ. For every $\varepsilon > 0$, there is an M such that if B has all side-lengths $\geq M$ and R is larger, "big enough" box in terms of B, the the following holds: If $v_1 + B, \ldots, v_m + B$ is a maximum-sized collection of disjoint B-translates in R and $U_1, \ldots, U_m \subseteq \mathbb{R}^n$ are convex and open, then*

$$|\Omega_R(\psi, U)| \geq \prod_{i=1}^{m} |\Omega_B(\psi(B_i)_\varepsilon)|,$$

where $U = \frac{1}{m}U_1 + \cdots + \frac{1}{m}U_m$ and $V_\varepsilon = \{x : \overline{B_\varepsilon(x)} \subseteq V\}$.



In fact, if $\omega \in A^R$ and $\omega|_{v_i+B} \in \Omega_{v_i+B}(\psi, (U_i)_\varepsilon)$ for all i, then $\omega \in \Omega_B(\psi, U)$.

Proof. We are assuming that $\frac{1}{|B|}\Psi_{v_i+B}(\omega_{v_i+B}) \in (U_i)_\varepsilon$ for all i. Consider

$$\begin{aligned}
\psi_R(\omega) &= \sum_{v+W\subseteq R} \psi(\omega_{v+W}) \\
&= \sum_i \sum_{v+W\subseteq v_i+B} \psi(\omega_{v+W}) + \underbrace{\sum_{\substack{v+W\not\subseteq v_i+B \\ \text{for any } i}} \psi(\omega_{v+W})}_{X}
\end{aligned}$$

$$
\begin{aligned}
&\in |B|\cdot (U_1)_\varepsilon + \cdots + |B|\cdot (U_m)_\varepsilon + X \\
&= |R|\frac{|B|}{|R|}\cdot ((U_1)_\varepsilon + \cdots + (U_m)_\varepsilon) + X \\
&= |R|\left(\frac{1}{m} + o_{R\uparrow \mathbb{Z}^d}(1)\right)((U_1)_\varepsilon + \cdots + (U_m)_\varepsilon) + X
\end{aligned}
$$

For big enough R,

$$
\begin{aligned}
&\subseteq |R|\left(\frac{U_1}{m} + \cdots + \frac{U_m}{m}\right)_{\varepsilon/2} + X \\
&= |B|U_{\varepsilon/2} + X
\end{aligned}
$$

Now estimate

$$
\begin{aligned}
|X| &= \left|\sum_{\substack{v+W\not\subseteq v_i+B \\ \text{for any } i}} \psi(\omega_{v+W})\right| \\
&\leq \|\psi\|_\infty \cdot \operatorname{diam}(W)\cdot \left(\sum_{v_i+B} |\partial(v_i+B)| + \left|R\setminus \bigcup_i (v_i+B)\right|\right) \\
&= O(1)\cdot (\underbrace{m\cdot|\partial B|} + o_{R\uparrow\mathbb{Z}^d}(|R|)),
\end{aligned}
$$

where this bracketed part will be small relative to $m|B| \leq |R|$ if B is big enough. So as $R\uparrow \mathbb{Z}^d$ and then $B\uparrow\mathbb{Z}^d$, we have

$$X = O(1)(o_{R\to\infty}(|R|) + o_{B\to\infty}(|R|)) = o_{R\to\infty, B\to\infty}(|R|).$$

So if B is big enough given ε and then R is big enough given B, then

$$\Psi_R(\omega) \in |R|U_{\varepsilon/2} + |X| \subseteq |R|\cdot U.$$

That is, $\omega\in\Omega_R(\psi, U)$. $\square$

Remark 1.1. ε, B, R did not depend on $U_1, \ldots, U_m$.

We can therefore restate the proposition as follows:

Corollary 1.1. *There exists a function* $\{boxes\}\to(0,\infty)$ *sending* $B\mapsto\varepsilon(B)$ *such that if*

- $\varepsilon(B)\downarrow 0$ *as* $B\uparrow\mathbb{Z}^d$,
- *For all* B, *if* R *is big enough in terms of* B *and* $U_1,\ldots,U_m$ *and* $v_1,\ldots,v_m$ *as before, then*

$$|\Omega_R(\psi, U)| \geq \prod_{i=1}^m |\Omega_B(\psi, (U_i)_{\varepsilon(B)})|.$$

Corollary 1.2. *There exists a function $s : \mathcal{U} = \{\text{open convex subsets of } \mathbb{R}^n\} \to [-\infty, \infty)$ such that for all $B_n \uparrow \mathbb{Z}^d$ and all $U \in \mathcal{U}$, we have*

$$\frac{1}{|B_n|} \log |\Omega_{B_n}(\psi, U_{2\varepsilon(B_n)})| = s(U) + o(1),$$

where

$$s(U) = \lim_n \frac{1}{|B_n|} \log |\Omega_{B_n}(\psi, U_{2\varepsilon(B_n)})|.$$

Proof. Let

$$s(U) := \sup_{\text{boxes } B} \frac{1}{|B|} \log |\Omega_B(\psi, U_{2\varepsilon(B)})|.$$

We will show that this agrees with the limit. The reason is that $\limsup_n \frac{1}{|B_n|}(\cdots) \leq \sup_{\text{boxes}} = s(U)$, so it is enough to show that $\liminf \geq s(U)$.

Let $B_n \uparrow \mathbb{Z}^d$ and fix a box B. Once B_n is big enough in terms of B, we can use the previous corollary to get

$$|\Omega_{B_n}(\psi, V)| \geq \prod_{i=1}^{m} |\Omega_B(\psi, (V_i)_{\varepsilon(B)})|$$

for all $V \in \mathcal{U}$, where m is the cardinality of a maximal packing of B-translates into B_n. Hence,

$$\frac{1}{|B_n|} \log |\Omega_{B_n}(\psi, V)| \geq \underbrace{\frac{m}{|B_n|}}_{1/|B|+o(1)} \log |\Omega_B(\psi, V_{\varepsilon(B)})|.$$

Apply this with $V = U_{2\varepsilon(B_n)}$. We get

$$\begin{aligned}\frac{1}{|B_n|} \log |\Omega_{B_n}(\psi, U_{2\varepsilon(B_n)})| &\geq \left(\frac{1}{|B|} + o(1)\right) \log |\Omega_B(\psi, U_{2\varepsilon(B)+\varepsilon(B)})| \\ &\geq \left(\frac{1}{|B|} + o(1)\right) \log |\Omega_B(\psi, U_{2\varepsilon(B)})|\end{aligned}$$

if n is big enough. Let $n \to \infty$ to get

$$\liminf_n \frac{1}{|B_n|} \log |\Omega_{B_n}(\psi, U_{2\varepsilon(B_n)})| \geq \frac{1}{|B|} \log |\Omega_B(\psi, U_{2\varepsilon(B)})|.$$

Take the sup over B and get $\lim_n = s(U)$. $\square$